Prof. Amador Martin-Pizarro — Albert-Ludwigs-Universität Freiburg
Exercises: Charlotte Amann-Bartnick — Winter term 2025/2026

Model Theory
Sheet 9
Deadline: 18.12.25, 2:30 pm.

Exercise 1 (12 points).

Given a structure $\mathcal{M}$ in a language $\mathcal{L}$ as well as a subset B of M, consider an indiscernible sequence $(a_n)_{n\in\mathbb{N}}$ over B in M.

a) Show that the sequence of 2-tuples $(a_{2n}, a_{2n+1})_{n\in\mathbb{N}}$ is also indiscernible over B.

b) Assume that there is an natural number n such that $\text{tp}^{\mathcal{M}}(a_0 \dots a_n/B)$ is algebraic. Show that the sequence is constant.

c) Describe all indiscernible sequences $(a_n)_{n\in\mathbb{N}}$ over $B = \emptyset$ in the $\mathbb{Q}$-vector space $\mathbb{Q}^{20}$ (as a structure in the language of $\mathbb{Q}$-vector spaces).

From now on let $B = \emptyset$ and suppose that we are given an indiscernible sequence in M indexed by $\mathbb{Z}$, that is $(a_n)_{n\in\mathbb{Z}}$ is indiscernible over $\emptyset$. Furthermore we assume that $\mathcal{L} = \{R\}$ consists solely of a binary relation symbol. Consider a tuple $\bar{c} = (c_1, \dots, c_m)$ from M.

d) Assume that $R^{\mathcal{M}}$ is an equivalence relation with infinitely many classes which are all infinite. Show that there is a finite subset I of $\mathbb{Z}$ such that the subsequence $(a_i)_{i\in\mathbb{Z}\setminus I}$ (with the induced ordering on $\mathbb{Z}\setminus I$) is indiscernible over $c_1, \dots, c_m$.

e) If $\mathcal{M} \models \text{DLO}$, give an example of an indiscernible sequence $(a_n)_{n\in\mathbb{Z}}$ over $\emptyset$ and a tuple $\bar{c}$ such that the conclusion from part d) is false. Moreover show that for every indiscernible sequence $(a_n)_{n\in\mathbb{Z}}$ there is N_0 from $\mathbb{Z}$ such that the subsequence $(a_n)_{n\geq N_0}$ is indiscernible over $c_1, \dots, c_m$.

f) We now assume that $\mathcal{M}$ is a countable random graph. Provide an example of sequence $(a_n)_{n\in\mathbb{Z}}$ which is indiscernible over $\emptyset$ and a tuple $\bar{c}$ such that the conclusion of part e) is false.

Exercise 2 (4 points).

A finite coloring of the natural number is a partition $\mathbb{N} = \bigcup_{i=1}^{n} C_i$ in disjoint subsets (or *colors*) C_i. Show that given any finite coloring of the natural numbers there is a monochromatic triple of the form $\{x, y, x+y\}$.

Hint: We color all 2-element subsets of $\mathbb{N}$ with n colors such that $\{a, b\}$ has color i if $|a-b|$ is in C_i.

(Please turn the page!)

Definition: A theory T in a given language $\mathcal{L}$ has definable Skolem functions if for every formula $\psi(x, \bar{y})$ with x a single variable there is a formula $\theta_\psi(\bar{y}, u)$ such that:

(i) $T \models \forall \bar{y} \left(\exists u \, \theta_\psi(\bar{y}, u) \wedge \forall u \, \forall v \, \big((\theta_\psi(\bar{y}, u) \wedge \theta_\psi(\bar{y}, v)) \rightarrow u = v \big) \right)$

(ii) $T \models \forall \bar{y} \, \big(\exists x \, \psi(x, \bar{y}) \rightarrow \exists u \, (\psi(u, \bar{y}) \wedge \theta_\psi(\bar{y}, u) \big)$.

In other words, the formula $\theta_\psi(\bar{y}, u)$ defines the graph of a function that yields for every choice of parameters $\bar{a}$ inside a given model of T an element in the definable set $\psi(x, \bar{a})$ if the latter is non-empty.

Exercise 3 (4 points)**.**

Suppose that $\mathcal{L}$ contains a binary relation symbol $<$ and consider an $\mathcal{L}$-structure $\mathcal{N} = (\mathbb{N}, <^{\mathcal{N}}, \ldots)$ on the natural numbers such that $<$ is interpreted as the usual order. Show that $T = \mathrm{Th}(\mathcal{N})$ has definable Skolem functions.